MATH4210: Financial Mathematics Tutorial 1

Xiangying Pang

The Chinese University of Hong Kong

xypang@math.cuhk.edu.hk

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Definition (Normal Distribution)

Given a real-valued random variable $X : \Omega \to \mathbb{R}$, it follows the normal distribution with parameters μ, σ if the probability density function (pdf) of X is given by $f : \mathbb{R} \to \mathbb{R}_+$:

$$\forall x \in \mathbb{R}, f(x) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$X \sim N(\mu, \sigma^2)$ means X follows the normal distribution with parameters μ, σ. $X \sim N(0,1)$ is called the standard normal distribution.

Exercise

Given $X \sim N(\mu, \sigma^2)$, compute the following:

(a). $\mathbb{E}(X), \mathbb{E}(X^2), Var(X)$; $= EX^2 - (EX)^2$

(b). $\mathbb{E}(|X|), \mathbb{E}((X-K)^+)$ with K fixed; $E(|X|) = E(X \cdot 1_{\{X \geq 0\}} + (-X) 1_{\{X<0\}})$

S(T) strike price

(c). $\mathbb{E}(e^{itX})$ for t fixed (Characteristic Function). $= E(X 1_{\{X\geq 0\}}) - E(\underbrace{X 1_{\{X<0\}}}_{X - X1_{\{X\geq 0\}}})$

Note that $f(t) := \mathbb{E}(e^{tX})$ is called the Moment Generating Function.

$= E(X1_{\{X\geq 0\}}) - EX + \cancel{E(X1_{\{X\geq 0\}})}$

(a) $EX = \int_{\mathbb{R}} x \cdot f(x)\,dx$

$= \int_{\mathbb{R}} x \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$

$\overset{y=x-\mu}{=\!=} \int_{\mathbb{R}} (y+\mu) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{y^2}{2\sigma^2}}\,dy$

$= \underbrace{\int_{-\infty}^{\infty} y \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}\,dy}_{=0} + \mu \cdot \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{y^2}{2\sigma^2}}\,dy}_{=1}$

odd function

$N(0,\sigma^2)$

$= \mu$

$EX^2 = \int_{\mathbb{R}} x^2 \cdot f(x)\,dx = \int_{\mathbb{R}} x^2 \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$

$\overset{y=x-\mu}{=\!=} \int_{\mathbb{R}} (y+\mu)^2 \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{y^2}{2\sigma^2}}\,dy$

$= \int_{\mathbb{R}} (y^2 + \mu^2 + 2\mu y) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{y^2}{2\sigma^2}}\,dy$

pdf   odd fun

$= \int_{\mathbb{R}} y^2 \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-y^2/2\sigma^2}\,dy + \mu^2 + 0$

$= \int_{\mathbb{R}} y \cdot \underbrace{y}_{dy^2/2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-y^2/2\sigma^2}\,dy + \mu^2$

$= \int_{\mathbb{R}} y \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{y^2}{2\sigma^2}} \cdot d\frac{y^2}{2} + \mu^2$

$= \int_{\mathbb{R}} y \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{1}{2} \cdot d\left(e^{-\frac{y^2}{2\sigma^2}}\right) \cdot (-2\sigma^2) + \mu^2$

$= (-\sigma^2) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} y \cdot d\left(e^{-y^2/2\sigma^2}\right) + \mu^2$

$= \frac{-\sigma^2}{\sqrt{2\pi\sigma^2}} \cdot \left(\underbrace{y \cdot e^{-y^2/2\sigma^2}\Big|_{-\infty}^{\infty}}_{0} - \underbrace{\int_{\mathbb{R}} e^{-y^2/2\sigma^2}\,dy}_{=\sqrt{2\pi\sigma^2}} \right) + \mu^2$

$N(0,\sigma^2)$

$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-y^2/2\sigma^2}\,dy = 1$

$= \frac{-\sigma^2}{\sqrt{2\pi\sigma^2}} \left(0 - \sqrt{2\pi\sigma^2}\right) + \mu^2$

$= \sigma^2 + \mu^2$

$Var(X) = EX^2 - (EX)^2$

$= \sigma^2 + \mu^2 - \mu^2 = \sigma^2$

$X \sim N(\mu, \sigma^2)$

$\mu \downarrow EX$, $\sigma^2 \downarrow VarX$

(b) $E((X-K)^+) = E((X-K)1_{\{X\geq K\}})$

$= \underbrace{E(X\cdot 1_{\{X\geq K\}})}_{①} - K\underbrace{E(1_{\{X\geq K\}})}_{②}$

①: $E(X 1_{\{X\geq K\}}) = \int_K^\infty x\cdot f(x)\,dx$

$= \int_K^\infty x\cdot \frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$

$\left(\Phi(t) := \int_{-\infty}^t \frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}}\,dx \text{ (cdf of } N(0,1))\right)$

$y = \frac{x-\mu}{\sigma}$

$\int_{\frac{K-\mu}{\sigma}}^\infty (\sigma y+\mu)\cdot\frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-\frac{y^2}{2}}\,\sigma\,dy$

$= \frac{\sigma^2}{\sqrt{2\pi\sigma^2}}\underbrace{\int_{\frac{K-\mu}{\sigma}}^\infty y\cdot e^{-\frac{y^2}{2}}\,dy}_{dy^2/2} + \frac{\mu\cdot\sigma}{\sigma\sqrt{2\pi\sigma^2}}\cdot\boxed{\int_{\frac{K-\mu}{\sigma}}^\infty e^{-\frac{y^2}{2}}\,dy}$

$\int_{\frac{K-\mu}{\sigma}}^\infty e^{-y^2/2}\,d\frac{y^2}{2}$

$= \frac{1}{2}\cdot e^{-\frac{y^2}{2}}\cdot(-2)\Big|_{\frac{K-\mu}{\sigma}}^\infty$

$= 0 + e^{-\frac{(K-\mu)^2}{2\sigma^2}}$

$\boxed{\int_{\frac{K-\mu}{\sigma}}^\infty e^{-\frac{y^2}{2}}\,dy} = \left(1-\int_{-\infty}^{\frac{K-\mu}{\sigma}} e^{-y^2/2}\,dy\right)$

$= 1-\Phi\left(\frac{K-\mu}{\sigma}\right)$

$= \frac{\sigma}{\sqrt{2\pi}} e^{-(K-\mu)^2/2\sigma^2} + \mu\cdot\left(1-\Phi\left(\frac{K-\mu}{\sigma}\right)\right)$

②: $E(1_{\{X\geq K\}})$

$= P(X\geq K)$

$(X = \sigma Y+\mu,\ Y\sim N(0,1))$

$= P(\sigma Y+\mu\geq K)$

$= P\left(Y\geq\frac{K-\mu}{\sigma}\right)$

$= 1-P\left(Y<\frac{K-\mu}{\sigma}\right)$

$= 1-\Phi\left(\frac{K-\mu}{\sigma}\right)$

(c) $E(e^{itX})$

$= \int_{\mathbb{R}} e^{itx}\cdot\frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$

$= \int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-\frac{[(x-\mu)^2-2\sigma^2 itx]}{2\sigma^2}}\,dx$

$(x-\mu)^2-2\sigma^2 itx$

$= x^2+\mu^2-2\mu x-2\sigma^2 itx$

$= (x-(\mu+\sigma^2 it))^2-(\mu+it\sigma^2)^2+\mu^2$

$= (x-(\mu+it\sigma^2))^2+t^2\sigma^4-2\mu it\sigma^2$

$-\frac{\sigma^2 t^2}{2}+\mu it \ldots (x-(\mu+it\sigma^2$

Exercise

Suppose $X_k \sim N\left(\mu_k, \sigma_k^2\right)$, $\lim \mu_k = \mu$, $\lim \sigma_k = \sigma$, and $X_k \to X$ in $\mathbb{L}^2$. Show X is a normal random variable with $\mathbb{E}[X] = \mu$ and $Var(X) = \sigma^2$.

$y = x - (\mu + it\sigma^2)$

$= e^{-\frac{\sigma^2 t^2}{2} + \mu i t} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2} dy$

$= e^{\mu i t - \frac{\sigma^2}{2} t^2}$

Review:

convergence in $\mathbb{L}^2$-norm: $\{X_n\}_{n=1}^{\infty}$ converges in $\mathbb{L}^2$-norm towards X, if $\lim_{n\to\infty} \mathbb{E}(|X_n - X|^2) = 0$.

convergence in probability:
$\{X_n\}_{n=1}^{\infty}$ converges in probability towards X if for all $\epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0$.

convergence in distribution: $\{X_n\}_{n=1}^{\infty}$ have cumulative distribution functions(cdf) $\{F_n\}_{n=1}^{\infty}$. If $\lim_{n\to\infty} F_n(x) = F(x)$ for every continuity point of F, then $\{X_n\}_{n=1}^{\infty}$ converge in distribution to a random variable X with cdf F.

relation: $X_n \xrightarrow{\mathbb{L}^2} X \Rightarrow X_n \xrightarrow{\mathbb{P}} X \Rightarrow X_n \xrightarrow{\mathcal{D}} X$

Lévy's convergence theorem: $\{\phi_n\}_{n=1}^{\infty}$ are characteristic functions of $\{X_n\}_{n=1}^{\infty}$. If $\phi_n(t) \to \phi(t)$ for $\forall t \in \mathbb{R}$, and $\phi(t)$ is a characteristic function of some random variable X, then X_n converges in distribution to the random variable X.

Since $X_k \sim N\left(\mu_k, \sigma_k^2\right)$, so the corresponding characteristic function $\phi_k(t) = e^{i\mu_k t - \frac{\sigma_k^2}{2}t^2}$.

It is obvious that $\phi_k(t) \to \phi(t) := e^{i\mu t - \frac{\sigma^2}{2}t^2}$ for $\forall t \in \mathbb{R}$, and $\phi(t)$ is a characteristic function of a random variable $Y \sim N(\mu, \sigma^2)$, then $X_k \xrightarrow{\mathcal{D}} Y$.

Since $X_n \xrightarrow{\mathbb{L}^2} X \Rightarrow X_n \xrightarrow{\mathcal{D}} X$ and the limit is unique, we have $X = Y \sim N(\mu, \sigma^2)$.

Exercise

Let $(Y_j)_{j\in\mathbb{N}}$ be a sequence of i.i.d. random variables. For any $j \in \mathbb{N}, \mathbb{P}(Y_j = \pm 1) = \frac{1}{2}$. Define for $n \in \mathbb{N}, X_n = \sum_{j=1}^{n} Y_j$. Show that $(X_n)_{n\in\mathbb{N}}$ is a martingale.

Solution:
In order to prove that $(X_n)_{n\in\mathbb{N}}$ is a martingale, we are going to verify by definition.
1. Fix $n \in \mathbb{N}$.

$$
\begin{aligned}
\mathbb{E}(|X_n|) &= \mathbb{E}(|\sum_{j=1}^{n} Y_j|) \\
&\leq \sum_{j=1}^{n} \mathbb{E}(|Y_j|) \\
&= n(1 * \frac{1}{2} + |-1| * \frac{1}{2}) \\
&= n < \infty
\end{aligned}
$$

2. Fix $n \in \mathbb{N}$, denote $\mathcal{F}_n = \sigma(X_0, ..., X_n)$.

$$\begin{aligned}
\mathbb{E}(X_{n+1}|\mathcal{F}_n) &= \mathbb{E}(X_n + Y_{n+1}|\mathcal{F}_n) \\
&= \mathbb{E}(X_n|\mathcal{F}_n) + \mathbb{E}(Y_{n+1}|\mathcal{F}_n) \\
&= X_n + \mathbb{E}(Y_{n+1}) \\
&= X_n
\end{aligned}$$

By 1 and 2, $(X_n)_{n\in\mathbb{N}}$ is a martingale.

Remark

It still works when $\mathbb{P}(Y_j = 2) = \frac{1}{3}$ and $\mathbb{P}(Y_j = -1) = \frac{2}{3}$. $(X_n)_{n\in\mathbb{N}}$ will still be a martingale as long as the expectation is 0 (Exercise!).